Journal of Mathematical Analysis and Applications **238,** 196–215 (1999)
Article ID jmaa.1999.6522, available online at http://www.idealibrary.com on IDEAL®

# Growing-Up Positive Solutions of Semilinear Elliptic Boundary Value Problems

Kenichiro Umezu



and

Kazuaki Taira

*Institute of Mathematics, University of Tsukuba, Tsukuba 305-8571, Japan*

*Submitted by Irena Lasiecka*

Received November 10, 1998

This paper is devoted to the study of the existence, uniqueness, and asymptotic behavior of positive solutions of a class of *degenerate* boundary value problems for semilinear second-order elliptic differential operators which originates from the so-called Yamabe problem in Riemannian geometry. Our approach is based on the super-sub-solution method adapted to the degenerate case. © 1999 Academic Press

*Key Words:* semilinear elliptic problem; growing-up positive solution; super-sub-solution method

## 1. INTRODUCTION AND RESULTS

Let $D$ be a bounded domain of Euclidean space $\mathbf{R}^N$, $N \geq 2$, with smooth boundary $\partial D$; its closure $\overline{D} = D \cup \partial D$ is an $N$-dimensional, compact smooth manifold with boundary. This paper is devoted to the study of the existence and uniqueness of positive solutions of the following

The research of the second author is partially supported by a Grant-in-Aid for General Scientific Research (no. 10440050), Ministry of Education, Science and Culture of Japan.



0022-247X/99 $30.00

semilinear elliptic boundary value problem:

$$
\begin{aligned}
-\Delta u &= \lambda u - h(x)u^p && \text{in } D,\\
Bu &:= a(x')\frac{\partial u}{\partial \mathbf{n}} + (1 - a(x'))u = 0 && \text{on } \partial D.
\end{aligned}
\tag{$*$}_\lambda
$$

Here

(1) $\Delta = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \cdots + \partial^2/\partial x_N^2$ is the usual Laplacian.

(2) $\lambda$ is a positive parameter.

(3) $h(x)$ is a real-valued function on the closure $\overline{D}$.

(4) $p > 1$.

(5) $\mathbf{n} = (n_1, n_2, \ldots, n_N)$ is the unit exterior normal to the boundary $\partial D$.

(6) $a(x')$ is a real-valued smooth function on the boundary $\partial D$.

A function $u(x) \in C^2(\overline{D})$ is called a *positive solution* of problem $(*)_\lambda$ if it satisfies problem $(*)_\lambda$ and is strictly positive everywhere in $D$.

It is worth pointing out here that the equation $-\Delta u - \lambda u + h(x)u^p = 0$ originates from the so-called Yamabe problem, which is a basic problem in Riemannian geometry if we take $p = (N+2)/(N-2) > 1$ for $N \geq 3$ (see [7, 8]).

Our fundamental conditions on the function $h(x)$ are the following:

$$h(x) \in C^\theta(\overline{D}), \qquad 0 < \theta < 1 \tag{H.1}$$

$$h(x) \geq 0 \qquad \text{on } \overline{D}. \tag{H.2}$$

We remark that Ouyang [9] and Korman and Ouyang [6] studied the case where the function $h(x)$ may change sign in $D$.

On the other hand, our boundary condition $B$ is a linear combination of the Dirichlet and Neumann conditions. It is easy to see that the boundary condition $B$ is nondegenerate (or coercive) if and only if either $a(x') \neq 0$ on $\partial D$ or $a(x') \equiv 0$ on $\partial D$. Ouyang [8] and del Pino [3] studied the Dirichlet and Neumann cases, while Fraile et al. [4] studied the general nondegenerate case. For further studies of semilinear elliptic problems, we refer to Alama and Tarantello [1], Amann [2], Gámez [5], and Pao [10].

In this paper we study problem $(*)_\lambda$ in the *degenerate* case; more precisely, our fundamental condition on the function $a(x')$ is the following:

$$0 \leq a(x') \leq 1 \qquad \text{on } \partial D. \tag{H.3}$$

Note that the so-called Lopatinskii–Shapiro complementary condition is violated at the points $x' \in \partial D$ where $a(x') = 0$.

To formulate our results, let $\lambda_1$ be the first eigenvalue of the linearized eigenvalue problem

$$\begin{aligned} -\Delta\varphi &= \lambda\varphi \qquad \text{in } D \\ B\varphi &= 0 \qquad \text{on } \partial D. \end{aligned} \tag{1.1}$$

It is known (see [11, Theorem 1]) that the first eigenvalue $\lambda_1$ is nonnegative and simple, and further that its associated eigenfunction $\varphi_1(x)$ can be chosen to be positive everywhere in $D$. By Green's formula, it is easy to see that a necessary condition on the parameter $\lambda$ for the existence of positive solutions of problem $(*)_\lambda$ is that $\lambda > \lambda_1$.

Conversely, if $h(x) > 0$ on $\overline{D}$, then Taira and Umezu [14] proved that problem $(*)_\lambda$ has a unique positive solution $u_\lambda(x) \in C^{2+\theta}(\overline{D})$ for each $\lambda > \lambda_1$. Furthermore, the solution $u_\lambda(x)$ grows up as $\lambda \to \infty$, that is, the maximum norm $\|u_\lambda\|_\infty$ on $\overline{D}$ tends to infinity as $\lambda \to \infty$.

This paper is concerned with the case where the function $h(x)$ may vanish in $D$. More precisely, we assume that

The zero set $D(h) = \{x \in \overline{D} : h(x) = 0\}$ of the function $h(x)$ is bounded away from the boundary $\partial D$, (H.4)

and denote by $D_0(h)$ its *interior*. Following del Pino [3], we introduce a critical value $\lambda_1(D_0(h))$ in the following way: Let $\mathscr{B}$ be the set of all open subsets of $D$ with smooth boundary. If $\Omega \in \mathscr{B}$, we denote by $\lambda_1(\Omega)$ the first eigenvalue of the Dirichlet problem,

$$\begin{aligned} -\Delta\varphi &= \lambda\varphi \qquad \text{in } \Omega, \\ \varphi &= 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

By the celebrated Rayleigh theorem, we know that the first eigenvalue $\lambda_1(\Omega)$ is given by the variational formula

$$\lambda_1(\Omega) = \inf\left\{\int_\Omega |\nabla u|^2\, dx : u \in H_0^1(\Omega), \int_\Omega u^2\, dx = 1\right\},$$

where $H_0^1(\Omega)$ is the closure of the space $C_0^\infty(\Omega)$ of smooth functions with compact support in $\Omega$ in the Sobolev space $H^1(\Omega)$. Then we let

$$\lambda_1(D_0(h)) = \sup\{\lambda_1(\Omega) : \Omega \in \mathscr{B}, D_0(h) \subset \Omega\}. \tag{1.2}$$

We understand $\lambda_1(D_0(h)) = \infty$ in the case where the set $D_0(h)$ is empty. Note that if the boundary $\partial D_0(h)$ is sufficiently regular, then the value

$\lambda_1(D_0(h))$ coincides with the first eigenvalue of the Dirichlet eigenvalue problem,

$$\begin{aligned} -\Delta\varphi &= \lambda\varphi && \text{in } D_0(h) \\ \varphi &= 0 && \text{on } \partial D_0(h). \end{aligned}$$

Our first result is the following existence and uniqueness theorem of positive solutions of problem $(*)_\lambda$:

THEOREM 1. *Assume that conditions* (H.1) *through* (H.4) *are satisfied. Then problem* $(*)_\lambda$ *has a unique positive solution* $u_\lambda(x) \in C^{2+\theta}(\overline{D})$ *for every* $\lambda_1 < \lambda < \lambda_1(D_0(h))$ *and no positive solution for all* $\lambda \geq \lambda_1(D_0(h))$. *Furthermore, the solution* $u_\lambda(x)$ *grows up as* $\lambda \uparrow \lambda_1(D_0(h))$, *that is, the maximum norm* $\|u_\lambda\|_\infty$ *tends to infinity as* $\lambda \uparrow \lambda_1(D_0(h))$.

Theorem 1 is a generalization of del Pino [3, Theorem 2], where the Dirichlet and Neumann conditions are treated, and it is proved by Taira and Umezu [13, Theorem 3] under the condition that the boundary $\partial D_0(h)$ is sufficiently regular.

Second, we study the asymptotic behavior of the unique positive solution $u_\lambda(x)$ as $\lambda \uparrow \lambda_1(D_0(h))$. To do so, we take a relatively compact, open subset $\Omega'$ of $D$ with smooth boundary $\partial\Omega'$, which satisfies the condition

$$\Omega' \supset \overline{D_0(h)}; \tag{1.3a}$$

the closure $\overline{\Omega'}$ consists of a *finite* number of connected components.

(1.3b)

Then we let

$$\begin{aligned} \Omega &= D \setminus \overline{\Omega'} \\ \Gamma &= \partial\Omega \cap D \end{aligned}$$

and introduce a nonnegative smooth function $\rho(x)$ defined on the closure $\overline{\Omega}$ such that

$$\rho(x) = \begin{cases} \inf\{|x-y| : y \in \Gamma\} & \text{on a tubular neighborhood of the topological boundary } \Gamma \text{ of } \Omega \text{ in } D; \\ 1 & \text{on a tubular neighborhood of the boundary } \partial D. \end{cases} \tag{1.4}$$

Now we can state our second result, which is a generalization of del Pino [3, Theorem 3] to the degenerate case:

THEOREM 2. *Assume that conditions* (H.1) *through* (H.4) *are satisfied. If* $\Omega'$ *is a relatively compact, open subset of* $D$ *with smooth boundary* $\partial\Omega'$ *which satisfies condition* (1.3) *and if* $\Omega = D \setminus \overline{\Omega'}$, *then, for any bounded subinterval* $I$ *of the interval* $(\lambda_1, \lambda_1(D_0(h)))$ *and any* $\alpha > 2/(p-1)$ *there exists a constant* $C > 0$ *such that we have*

$$\sup_{\lambda \in I} u_\lambda(x) \leq C\rho(x)^{-\alpha}, \qquad x \in \Omega. \tag{1.5}$$

*Furthermore, if the interior* $D_0(h)$ *is connected and nonempty, then we have, for any compact subset* $K$ *of* $D_0(h)$,

$$\inf_{x \in K} u_\lambda(x) \to \infty \qquad \text{as } \lambda \uparrow \lambda_1(D_0(h)). \tag{1.6}$$

Rephrased, Theorem 2 asserts that the more the exponent $p$ increases, the more mildly the solution $u_\lambda(x)$ behaves; while the more the set $D_0(h)$ enlarges, the wilder the solution $u_\lambda(x)$ is.

If the set $D_0(h)$ is equal to the unit open ball in $\mathbf{R}^N$, then we can give a precise description of the growing-up rate of the solution $u_\lambda(x)$ in $D_0(h)$ (see Theorem 5.1). For a similar description, we refer to del Pino [3, Remark 1], where the growing-up rate in the set $\overline{D} \setminus D(h)$ is given for a smooth function $h(x)$.

Finally we discuss the behavior of the solution $u_\lambda(x)$ as $\lambda \uparrow \lambda_1(D_0(h))$ in the case where the zero set $D(h)$ is nonempty but its interior $D_0(h)$ is empty. Then Theorem 1 tells us that there exists a unique positive solution $u_\lambda(x)$ of problem $(*)_\lambda$ for each $\lambda > \lambda_1$ and that the maximum norm $\|u_\lambda\|_\infty$ tends to infinity as $\lambda \to \infty$.

Our third result generalizes assertion (1.6) of Theorem 2 to the case where the interior $D_0(h)$ is empty:

THEOREM 3. *Assume that conditions* (H.1) *through* (H.4) *are satisfied, and further that there exists a sequence* $\{\Omega_j\}_{j=1}^\infty$ *of relatively compact, open subsets of* $D$ *with smooth boundary such that the* $\Omega_j$ *contain the zero set* $D(h)$ *and satisfy the condition*

$$\lim_{j \to \infty} |\Omega_j| = 0,$$

*where* $|\cdot|$ *denotes the Lebesgue measure of a measurable set of* $\mathbf{R}^N$. *Then problem* $(*)_\lambda$ *has a unique positive solution* $u_\lambda(x)$ *for each* $\lambda > \lambda_1$ *which tends to infinity as* $\lambda \to \infty$, *uniformly with respect to* $x \in K$ *for any compact subset* $K$ *of* $D$.

EXAMPLE 1. If the zero set $D(h)$ consists of *finitely* many points in $D$, then Theorem 3 applies.

EXAMPLE 2. If the zero set $D(h)$ consists of finitely many connected components of dimension $m$ with $1 \le m \le N-1$, then Theorem 3 applies.

The rest of this paper is organized as follows.

In Section 2 we prove Theorem 1 by using the super-sub-solution method and comparison arguments with the Dirichlet and Neumann conditions. Section 3 is devoted to the proof of Theorem 2. Our approach is based on a modification of the variational technique of del Pino [3] adapted to the degenerate case. In Section 4 we prove Theorem 3. The essential step in the proof is how to construct a super-solution of problem $(*)_\lambda$ to prove the existence of a positive solution, while we construct a good sub-solution to study the behavior of the positive solution, by making use of the eigenfunction $\varphi_1(x)$ of problem (1.1). In Section 5 we consider the growing-up rate of the unique positive solution $u_\lambda(x)$ in the case where the interior $D_0(h)$ is the unit open ball in $\mathbf{R}^N$ and the function $h(x)$ satisfies a growth condition near the boundary $\partial D_0(h)$ (Theorem 5.1). To give a precise description of the growing-up rate of the solution $u_\lambda(x)$, we transpose problem $(*)_\lambda$ into an equivalent fixed-point equation $(**)_\lambda$ for the resolvent $K$, and then apply the super-sub-solution method (Theorem 5.2).

## 2. PROOF OF THEOREM 1

In this section we prove Theorem 1 by using the super-sub-solution method and comparison arguments with the Dirichlet and Neumann conditions. By the work of Taira and Umezu [13], we know that the problem $(*)_\lambda$ has at most one positive solution for every $\lambda > \lambda_1$.

(I) First we prove that problem $(*)_\lambda$ has a positive solution $u_\lambda(x)$ for every $\lambda_1 < \lambda < \lambda_1(D_0(h))$, by using the super-sub-solution method.

Let $f(x,t)$ be a real-valued, Hölder continuous function with exponent $0 < \theta < 1$ on $\overline{D} \times [0,r]$ for any $r > 0$, and satisfy the following *slope condition* or *one-sided Lipschitz condition* (cf. [2, 10]): For any $r > 0$, there exists a constant $L > 0$ such that

$$f(x,t) - f(x,s) > -L(t-s), \qquad x \in \overline{D}, \quad 0 \le s < t \le r.$$

Now we consider the solvability of the semilinear elliptic problem

$$
\begin{aligned}
-\Delta u &= f(x,u) && \text{in } D \\
Bu &= a(x')\frac{\partial u}{\partial \mathbf{n}} + (1-a(x'))u = 0 && \text{on } \partial D.
\end{aligned}
\tag{2.1}
$$

A nonnegative function $\phi(x) \in C^2(\overline{D})$ is called a *sub-solution* of problem (2.1) if it satisfies the conditions

$$
\begin{aligned}
-\Delta \phi &\leq f(x,\phi) && \text{in } D \\
B\phi &\leq 0 && \text{on } \partial D.
\end{aligned}
$$

Similarly, a nonnegative function $\psi(x) \in C^2(\overline{D})$ is called a *super-solution* of problem (2.1) if it satisfies the conditions

$$
\begin{aligned}
-\Delta \psi &\geq f(x,\psi) && \text{in } D, \\
B\psi &\geq 0 && \text{in } \partial D.
\end{aligned}
$$

The next theorem [13, Theorem 1] plays a fundamental role in the construction of positive solutions of problem (2.1) (cf. [2, Theorem 9.4], [10, Theorems 3.2.1 and 3.2.2] for the nondegenerate case):

THEOREM 2.1. *Assume that condition* (H.3) *is satisfied. If there exist a sub-solution* $\phi(x)$ *and a super-solution* $\psi(x)$ *of problem* (2.1) *such that* $\phi(x) \leq \psi(x)$ *on* $\overline{D}$, *then problem* (2.1) *has a solution* $u(x) \in C^{2+\theta}(\overline{D})$ *such that* $\phi(x) \leq u(x) \leq \psi(x)$ *on* $\overline{D}$.

(Ia) We construct a super-solution of problem $(*)_\lambda$ for each $\lambda_1 < \lambda < \lambda_1(D_0(h))$, by using the following existence result for the Neumann problem due to del Pino [3] (see Ouyang [8] for the case where $\partial D_0(h)$ is sufficiently smooth):

THEOREM 2.2. *Assume that conditions* (H.1), (H.2), *and* (H.4) *are satisfied. Then the homogeneous Neumann problem*

$$
\begin{aligned}
-\Delta v &= \lambda v - h(x)v^p && \textit{in } D \\
\frac{\partial v}{\partial \mathbf{n}} &= 0 && \textit{on } \partial D
\end{aligned}
\tag{2.2}
$$

*has a unique positive solution* $v_\lambda(x) \in C^{2+\theta}(\overline{D})$ *for each* $0 < \lambda < \lambda_1(D_0(h))$.

Let $\psi_\lambda(x)$ be a unique positive solution of problem (2.2) for $0 < \lambda < \lambda_1(D_0(h))$. Then it follows that the function $\psi_\lambda$ is a super-solution of problem $(*)_\lambda$, since we have

$$
B\psi_\lambda = a(x')\frac{\partial \psi_\lambda}{\partial \mathbf{n}} + (1-a(x'))\psi_\lambda = (1-a(x'))\psi_\lambda \geq 0 \qquad \text{on } \partial D.
$$

(Ib) Next we construct a sub-solution of problem $(*)_\lambda$. Let $\varphi_1(x) \in C^\infty(\overline{D})$ be the positive eigenfunction corresponding to the first eigenvalue $\lambda_1$ of problem (1.1), normalized as $\|\varphi_1\|_\infty = 1$. If $\lambda > \lambda_1$, then we have, for $\varepsilon > 0$,

$$-\Delta(\varepsilon\varphi_1) - \lambda\varepsilon\varphi_1 + h(x)(\varepsilon\varphi_1)^p \leq \big(\|h\|_\infty \varepsilon^{p-1} - (\lambda - \lambda_1)\big)\varepsilon\varphi_1 \qquad \text{in } D.$$

This proves that the function $\varepsilon_\lambda \varphi_1(x)$ is a sub-solution of problem $(*)_\lambda$ if $\varepsilon_\lambda$ is sufficiently small.

(Ic) By [12, Lemma 2.1], we see that the functions $\psi_\lambda(x)$ and $\varphi_1(x)$ are comparable. This implies that if $\varepsilon_\lambda$ is sufficiently small, then it follows that $\varepsilon_\lambda \varphi_1(x) \leq \psi_\lambda(x)$ on $\overline{D}$. Therefore, by applying Theorem 2.1 we can find a positive solution $u_\lambda(x)$ of problem $(*)_\lambda$ for every $\lambda_1 < \lambda < \lambda_1(D_0(h))$ such that

$$\varepsilon_\lambda \varphi_1(x) \leq u_\lambda(x) \leq \psi_\lambda(x) \qquad \text{on } \overline{D}.$$

(II) Second, we prove a nonexistence result for all $\lambda \geq \lambda_1(D_0(h))$. To do so, we need the following existence and nonexistence results for the Dirichlet problem due to del Pino [3] (see Ouyang [8] for the case where $\partial D_0(h)$ is sufficiently smooth):

THEOREM 2.3. *Assume that conditions* (H.1), (H.2), *and* (H.4) *are satisfied. Then the homogeneous Dirichlet problem*

$$\begin{aligned} -\Delta w &= \lambda w - h(x)w^p && \text{in } D \\ w &= 0 && \text{on } \partial D \end{aligned} \tag{2.3}$$

*has a unique positive solution $w_\lambda(x) \in C^{2+\theta}(\overline{D})$ for each $\lambda_1(D) < \lambda < \lambda_1(D_0(h))$, and it has no positive solution for all $\lambda \geq \lambda_1(D_0(h))$. Here $\lambda_1(D)$ is the first eigenvalue of the Dirichlet eigenvalue problem*

$$\begin{aligned} -\Delta u &= \lambda u && \text{in } D \\ u &= 0 && \text{on } \partial D. \end{aligned}$$

Now assume to the contrary that problem $(*)_\lambda$ has a positive solution $u_\lambda(x)$ for some $\lambda \geq \lambda_1(D_0(h))$. Then it follows that the function $u_\lambda(x)$ is a super-solution of problem (2.3), since we have

$$u_\lambda \geq 0 \qquad \text{on } \partial D.$$

On the other hand, if $\varphi_0(x)$ is a positive eigenfunction corresponding to the first eigenvalue $\lambda_1(D)$, then it is easy to verify that the function $\varepsilon_\lambda \varphi_0(x)$ is a sub-solution of problem (2.3) for $\varepsilon_\lambda$ sufficiently small, since we have $\lambda > \lambda_1(D)$.

Therefore, by applying Theorem 2.1 to the Dirichlet case ($a(x') \equiv 0$ on $\partial D$) we can find a positive solution $w_\lambda(x) \in C^{2+\theta}(\overline{D})$ of problem (2.3) for $\lambda \geq \lambda_1(D_0(h))$ such that

$$\varepsilon_\lambda \varphi_0(x) \leq w_\lambda(x) \leq u_\lambda(x) \qquad \text{on } \overline{D}.$$

However, this contradicts Theorem 2.3.

(III) Finally, since we have proved the existence and nonexistence results for problem $(*)_\lambda$, we can prove just as in [13] that the maximum norm $\|u_\lambda\|_\infty$ tends to infinity as $\lambda \uparrow \lambda_1(D_0(h))$.

The proof of Theorem 1 is now complete. ∎

## 3. PROOF OF THEOREM 2

This section is devoted to the proof of Theorem 2. Our approach is based on a modification of the variational technique of del Pino [3] adapted to the degenerate case. The proof is divided into four steps.

(I) We introduce a nonnegative smooth function $\rho(x) \in C^\infty(\overline{\Omega})$ defined by formula (1.4) and consider a function

$$v(x) = C\rho(x)^{-\alpha}, \qquad \alpha > 2/(p-1),$$

where $C$ is a positive constant to be chosen later on. Then we have, by a direct computation,

$$-\Delta v = C\left(\alpha\rho^{-\alpha-1}\Delta\rho - \alpha(\alpha+1)\rho^{-\alpha-2}|\nabla\rho|^2\right).$$

Since $\rho(x) = 1$ in a tubular neighborhood of $\partial D$, by integration by parts it follows that we have, for all nonnegative functions $\varphi(x) \in C^1(\Omega)$ having support away from $\Gamma = \partial\Omega \cap D$,

$$\begin{aligned}\int_\Omega \nabla v \cdot \nabla\varphi \, dx &= -\int_\Omega \Delta v \cdot \varphi \, dx + \int_{\partial\Omega} \frac{\partial v}{\partial \mathbf{n}} \varphi \, d\sigma \\ &= C\int_\Omega \left(\alpha\rho^{-\alpha-1}\Delta\rho - \alpha(\alpha+1)\rho^{-\alpha-2}|\nabla\rho|^2\right)\varphi \, dx, \quad (3.1)\end{aligned}$$

where $d\sigma$ is the surface element on $\partial\Omega$.

On the other hand, we find that any positive solution $u(x)$ of problem $(*)_\lambda$ satisfies the formula

$$\int_\Omega (\nabla u \cdot \nabla \varphi + (hu^p - \lambda u)\varphi)\, dx = \int_{\partial\Omega} \frac{\partial u}{\partial \mathbf{n}} \varphi \, d\sigma. \tag{3.2}$$

However, since we have

$$Bu = a(x') \frac{\partial u}{\partial \mathbf{n}} + (1 - a(x'))u = 0,$$

it follows that

$$\begin{aligned} u(x') &= 0 && \text{if } a(x') = 0 \\ \frac{\partial u}{\partial \mathbf{n}}(x') &= -\frac{1 - a(x')}{a(x')} u(x') \le 0 && \text{if } a(x') > 0, \end{aligned}$$

so that

$$\frac{\partial u}{\partial \mathbf{n}} \le 0 \qquad \text{on } \partial D.$$

Therefore, combining inequalities (3.1) and (3.2) we have, for all non-negative functions $\varphi \in C^1(\Omega)$ having support away from $\Gamma$,

$$\begin{aligned} &\int_\Omega (\nabla(u - v) \cdot \nabla\varphi + (hu^p - \lambda u)\varphi)\, dx \\ &\quad = C\int_\Omega \left(-\alpha\rho^{-\alpha-1}\Delta\rho + \alpha(\alpha+1)\rho^{-\alpha-2}|\nabla\rho|^2\right)\varphi\, dx + \int_{\partial\Omega} \frac{\partial u}{\partial \mathbf{n}} \varphi\, d\sigma \\ &\quad \le C\int_\Omega \left(-\alpha\rho^{-\alpha-1}\Delta\rho + \alpha(\alpha+1)\rho^{-\alpha-2}|\nabla\rho|^2\right)\varphi\, dx. \end{aligned} \tag{3.3}$$

If we let

$$\underline{h} = \tfrac{1}{2} \inf_{x\in\Omega} h(x),$$

then we obtain from inequality (3.3) that

$$\begin{aligned} &\int_\Omega \left(\nabla(u - v) \cdot \nabla\varphi + (hu^p - \lambda u - \underline{h}v^p)\varphi\right) dx \\ &\quad \le C|\Omega|\, \|\varphi\|_{\infty,\overline{\Omega}} \|\rho\|_{\infty,\overline{\Omega}}^{-\alpha p} \Big(\alpha \|\rho\|_{\infty,\overline{\Omega}}^{-\alpha-1+\alpha p} \|\Delta\rho\|_{\infty,\overline{\Omega}} \\ &\qquad + \alpha(\alpha+1)\|\rho\|_{\infty,\overline{\Omega}}^{-\alpha-2+\alpha p} \|\,|\nabla\rho|\,\|_{\infty,\overline{\Omega}} - \underline{h}C^{p-1}\Big), \end{aligned} \tag{3.4}$$

where $\|\cdot\|_{\infty,\overline{\Omega}}$ is the maximum norm of $C(\overline{\Omega})$. Since $\underline{h} > 0$ and $-\alpha - 2 + \alpha p > 0$, it follows from inequality (3.4) that

$$\int_{\Omega} \left( \nabla(u - v) \cdot \nabla\varphi + \left( hu^p - \lambda u - \underline{h}v^p \right) \varphi \right) dx \leq 0, \tag{3.5}$$

if we take the constant $C$ (independent of $\lambda$) so large that

$$C^{p-1} > \frac{\alpha}{\underline{h}} \left( \|\rho\|_{\infty,\overline{\Omega}} \|\Delta\rho\|_{\infty,\overline{\Omega}} + (\alpha + 1) \| |\nabla\rho| \|_{\infty,\overline{\Omega}} \right) \|\rho\|_{\infty,\overline{\Omega}}^{-\alpha-2+\alpha p}. \tag{3.6}$$

We remark that formula (3.5) remains valid for all nonnegative functions $\varphi$ in the Sobolev space $H^1(\Omega)$ having support away from $\Gamma$.

(II) Let $I$ be a bounded subinterval of the interval $(\lambda_1, \lambda_1(D_0(h)))$, and let $u_\lambda(x)$ be a positive solution of problem $(*)_\lambda$ for $\lambda \in I$. Then we have the assertion

$$x \in \Omega \quad \text{and} \quad u_\lambda(x) \geq C\rho(x)^{-\alpha} \Rightarrow h(x)u_\lambda(x)^p - \lambda u_\lambda(x) \geq \underline{h}u_\lambda(x)^p, \tag{3.7}$$

if we take a constant $C > 0$ sufficiently large, independent of $\lambda \in I$.

Indeed, if we let

$$\overline{\lambda} = \sup_{\lambda \in I} \lambda,$$

then it follows that

$$\begin{aligned} h(x)u_\lambda(x)^p - \lambda u_\lambda(x) - \underline{h}u_\lambda(x)^p &\geq \underline{h}u_\lambda(x)^p - \overline{\lambda}u_\lambda(x) \\ &= u(x)\left(\underline{h}u_\lambda(x)^{p-1} - \overline{\lambda}\right) \\ &\geq u(x)\left(\underline{h}C^{p-1}\rho(x)^{-\alpha(p-1)} - \overline{\lambda}\right) \\ &\geq u(x)\left(\underline{h}C^{p-1}\|\rho\|_{\infty,\overline{\Omega}}^{-\alpha(p-1)} - \overline{\lambda}\right) \\ &> 0, \end{aligned}$$

if we take the constant $C > 0$ so large that

$$C^{p-1} > \frac{\overline{\lambda}}{\underline{h}} \|\rho\|_{\infty,\overline{\Omega}}^{\alpha(p-1)}. \tag{3.8}$$

(III) To prove assertion (1.5), assume to the contrary that one can find a bounded interval $I_0 \subset (\lambda_1, \lambda_1(D_0(h)))$ such that, for any constant

$C > 0$ there exist a parameter $\lambda_0 \in I_0$ and a point $x_0 \in \Omega$ such that the unique positive solution $u_{\lambda_0}(x)$ of problem $(*)_{\lambda_0}$ satisfies the inequality

$$u_{\lambda_0}(x_0) > C\rho(x_0)^{-\alpha}.$$

We choose a large constant $C$ satisfying inequalities (3.6) and (3.8) with $I := I_0$, and let

$$\begin{aligned} u_0(x) &= u_{\lambda_0}(x), \\ v_0(x) &= C\rho(x)^{-\alpha}, \\ w_0(x) &= \max\{u_0(x) - v_0(x), 0\}. \end{aligned}$$

Then it follows that the function $w_0(x)$ belongs to the Sobolev space $H^1(\Omega)$ having support away from $\Gamma$, since we have, for all $x$ in a tubular neighborhood of $\Gamma$ in $\Omega$,

$$u_0(x) < v_0(x).$$

Hence, applying inequality (3.5) to the functions $u := u_0$, $v := v_0$, and $\varphi := w_0$, we obtain

$$\int_\Omega \left(\nabla(u_0 - v_0) \cdot \nabla w_0 + \left(hu_0^p - \lambda_0 u_0 - \underline{h} v_0^p\right) w_0\right) dx \leq 0,$$

or equivalently,

$$\int_{\operatorname{supp} w_0} \left(\nabla(u_0 - v_0) \cdot \nabla w_0 + \left(hu_0^p - \lambda_0 u_0 - \underline{h} v_0^p\right) w_0\right) dx \leq 0.$$

Furthermore, in view of assertion (3.7) this implies that

$$\int_{\operatorname{supp} w_0} \left(\nabla(u_0 - v_0) \cdot \nabla w_0 + \underline{h}\left(u_0^p - v_0^p\right) w_0\right) dx \leq 0.$$

Therefore we conclude that

$$\begin{aligned} 0 &< \int_{\operatorname{supp} w_0} \underline{h}\left(u_0^p - v_0^p\right) w_0 \, dx \\ &\leq \int_{\operatorname{supp} w_0} \left(\nabla(u_0 - v_0) \cdot \nabla w_0 + \underline{h}\left(u_0^p - v_0^p\right) w_0\right) dx \\ &\leq 0, \end{aligned}$$

since $w_0 = u_0 - v_0$ on $\operatorname{supp} w_0$ and the Lebesgue measure of $\operatorname{supp} w_0$ is positive. This contradiction proves assertion (1.5).

(IV) Finally it remains to prove assertion (1.6). Let $u_\lambda(x)$ be a positive solution of problem $(*)_\lambda$ for any $\lambda_1(D) < \lambda < \lambda_1(D_0(h))$, and let $\varphi_0(x)$ be a positive eigenfunction corresponding to the first eigenvalue $\lambda_1(D)$ of problem (2.4). Then, just as in step (II) of the proof of Theorem 1, we find that the function $u_\lambda(x)$ is a super-solution of problem (2.3) and, furthermore, that the function $\varepsilon_\lambda \varphi_0(x)$ is a sub-solution of problem (2.3) if $\varepsilon_\lambda$ is sufficiently small.

Therefore, by applying Theorem 2.1 to the Dirichlet case ($a(x') \equiv 0$ on $\partial D$) we can find a positive solution $w_\lambda(x) \in C^{2+\theta}(\overline{D})$ of problem (2.3) such that

$$\varepsilon_\lambda \varphi_0(x) \le w_\lambda(x) \le u_\lambda(x), \qquad x \in \overline{D}. \tag{3.9}$$

However, we know from part (ii) of [3, Theorem 3] that assertion (1.6) holds for the function $w_\lambda(x)$, that is,

$$\inf_{x \in K} w_\lambda(x) \to \infty \qquad \text{as } \lambda \uparrow \lambda_1(D_0(h)).$$

In view of inequalities (3.9), it follows that assertion (1.6) holds also for the solution $u_\lambda(x)$ of problem $(*)_\lambda$.

Now the proof of Theorem 2 is complete. ∎

## 4. PROOF OF THEOREM 3

In this section we prove Theorem 3. The essential step in the proof is how to construct a super-solution of problem $(*)_\lambda$ to prove the existence of a positive solution, while we construct a good sub-solution to study the behavior of the positive solution, by making use of the eigenfunction $\varphi_1(x)$ of problem (1.1).

(I) Now take the positive eigenfunction $\varphi_1(x)$ corresponding to the first eigenvalue $\lambda_1$ of problem (1.1) such that $\|\varphi_1\|_\infty = 1$ and let

$$v(x) = \left( \frac{\lambda - \lambda_1}{\|h\|_\infty} \right)^{1/(p-1)} \varphi_1(x), \qquad \lambda > \lambda_1.$$

Then we have

$$-\Delta v - \lambda v + h(x) v^p \le 0 \qquad \text{in } D.$$

This implies that $v(x)$ is a sub-solution of problem $(*)_\lambda$.

(II) Next we construct a super-solution of problem $(*)_\lambda$. Since the function $h(x)$ satisfies conditions (H.1), (H.2), and (H.4), for each $\lambda > \lambda_1$ we can choose a nonnegative function $\tilde{h}(x) \in C^\theta(\overline{D})$ such that

1. The zero set $D(\tilde{h})$ of the function $\tilde{h}(x)$ is bounded away from $\partial D$.
2. The interior $D_0(\tilde{h})$ of $D(\tilde{h})$ is not empty.
3. The interior $D_0(\tilde{h})$ contains the zero set $D(h)$ of the function $h(x)$.
4. $\tilde{h}(x) \le h(x)$, $x \in \overline{D}$.
5. $\lambda < \lambda_1(D_0(\tilde{h}))$.

Here we have used the fact that the value $\lambda_1(D_0(h))$ defined by formula (1.2) tends to infinity as the Lebesgue measure $|D_0(h)|$ goes to zero.

Now we consider the following boundary value problem:

$$\begin{aligned} -\Delta u &= \lambda u - \tilde{h}(x)u^p && \text{in } D \\ Bu &= 0 && \text{on } \partial D. \end{aligned} \tag{4.1}$$

Theorem 1 tells us that problem (4.1) has a unique positive solution $w(x) \in C^{2+\theta}(\overline{D})$. Then it follows that the function $Cw(x)$ is a super-solution of problem $(*)_\lambda$ for all $C \ge 1$. Indeed, we have

$$-\Delta(Cw) - \lambda Cw + h(x)(Cw)^p = Cw^p\big(C^{p-1}h(x) - \tilde{h}(x)\big) \ge 0 \qquad \text{in } D,$$

since $h(x) \ge \tilde{h}(x)$ on $\overline{D}$ and $p > 1$.

(III) Applying Theorem 2.1 to the sub-solution $v(x)$ and the super-solution $Cw(x)$ for $C$ sufficiently large, we can find a solution $u_\lambda(x)$ of problem $(*)_\lambda$ such that

$$v(x) = \left(\frac{\lambda - \lambda_1}{\|h\|_\infty}\right)^{1/(p-1)} \varphi_1(x) \le u_\lambda(x) \le Cw(x) \qquad \text{on } \overline{D}.$$

This proves that the solution $u_\lambda(x)$ tends to infinity as $\lambda \to \infty$, uniformly with respect to $x \in K$ for any compact subset $K$ of $D$, since we have $\varphi_1(x) > 0$ in $D$.

The proof of Theorem 3 is now complete. ∎

## 5. GROWING-UP RATE OF POSITIVE SOLUTIONS

In this section we study the growing-up rate of the unique positive solution $u_\lambda(x)$ of problem $(*)_\lambda$ as $\lambda \uparrow \lambda_1(D_0(h))$ under some further restrictions on the function $h(x)$, which generalizes assertion (1.6) of

Theorem 2 (Theorem 5.1). To give a precise description of the growing-up rate of the solution $u_\lambda(x)$, we transpose problem $(*)_\lambda$ into an equivalent fixed-point equation $(**)_\lambda$ for the resolvent $K$, and then apply the super-sub-solution method (Theorem 5.2).

We assume that the zero set $D(h)$ of the function $h(x)$ is given by the formula

$$D(h) = \{x \in \mathbf{R}^N : |x| \le 1\}, \tag{5.1}$$

and we let

$$D_1 := D_0(h) = \{x \in \mathbf{R}^N : |x| < 1\}$$
$$D_r := \{x \in \mathbf{R}^N : |x| < r\}, \qquad r > 1.$$

Concerning the growth rate of $h(x)$ near $\partial D_0(h)$, we assume that there exist constants $\sigma > 1$, $C_1 > 0$, and $\delta_0 > 0$ such that

$$\sup_{x \in D_r} h(x) \le C_1(r-1)^\sigma, \qquad r \in [1, 1+\delta_0]. \tag{5.2}$$

EXAMPLE 5.1. If $h(x)$ is a function in $C^1(\mathbf{R}^N)$ given by the formula

$$h(x) = \begin{cases} 0 & |x| < 1 \\ (|x|-1)^2 & |x| \ge 1, \end{cases}$$

then $h(x)$ satisfies conditions (5.1) and (5.2) with $\sigma = 2$.

Now let $\phi_r(x)$ be a positive eigenfunction associated with the first eigenvalue $\lambda_1(D_r)$ of the Dirichlet problem

$$\begin{aligned} -\Delta\varphi &= \mu\varphi && \text{in } D_r \\ \varphi &= 0 && \text{on } \partial D_r, \end{aligned} \tag{5.3}$$

where the eigenfunction $\phi_r(x)$ is normalized as $\|\phi_r\|_{\infty, \overline{D_r}} = 1$.

The next theorem gives a precise description of the growing-up rate of the positive solution $u_\lambda(x)$ in assertion (1.6) of Theorem 2:

THEOREM 5.1. *Assume that the function $h(x)$ satisfies conditions* (5.1) *and* (5.2) *and that its zero set $D(h)$ is bounded away from the boundary $\partial D$. If $u_\lambda(x)$ is a unique positive solution of problem $(*)_\lambda$ for $\lambda_1 < \lambda < \lambda_1(D_0(h))$, then, for any compact subset $K$ of $D_0(h)$ there exists a constant $C > 0$, independent of $\lambda$, such that*

$$u_\lambda(x) \ge C(\lambda_1(D_0(h)) - \lambda)^{(1-\sigma)/(p-1)}\phi_1(x) \qquad \textit{for all } x \in K, \tag{5.4}$$

*for $\lambda$ sufficiently close to $\lambda_1(D_0(h))$.*

*Proof.* The proof is divided into four steps.

(I) First, we transpose problem $(*)_\lambda$ into an equivalent fixed-point equation for the resolvent of the linearized boundary value problem. By using [13, Theorem 1.1] for a given constant d $> 0$ we can associate with the boundary value problem

$$\begin{aligned} (-\Delta + d)u &= f \qquad \text{in } D \\ Bu &= 0 \qquad \text{on } \partial D \end{aligned} \tag{5.5}$$

a linear operator

$$K_d\colon C^\theta(\overline{D}) \to C^{2+\theta}(\overline{D})$$

in the following way. For any function $f \in C^\theta(\overline{D})$, the function $u = K_d f \in C^{2+\theta}(\overline{D})$ is the unique solution of problem (5.5). Then it is easy to verify that the operator $K$ is uniquely extended to a strictly positive, compact linear operator $K$ from the ordered Banach space $C(\overline{D})$ into itself [12, Lemma 2.1]. Furthermore, we find that problem $(*)_\lambda$ is equivalent to a nonlinear operator equation,

$$u = K_d((\lambda + d)u - hu^p) \qquad \text{in } C(\overline{D}). \tag*{$(**)_\lambda$}$$

(II) Second, we apply the super-sub-solution method to solve equation $(**)_\lambda$.

A nonnegative function $\phi(x) \in C(\overline{D})$ is said to be a *super-solution* of equation $(**)_\lambda$ if it satisfies the condition

$$\phi(x) \geq K_d((\lambda + d)\phi - h\phi^p)(x) \qquad \text{for } x \in \overline{D}.$$

Similarly, a nonnegative function $\psi(x) \in C(\overline{D})$ is said to be a *sub-solution* of equation $(**)_\lambda$ if it satisfies the condition

$$\psi(x) \leq K_d((\lambda + d)\psi - h\psi^p)(x) \qquad \text{for } x \in \overline{D}.$$

The next existence theorem for problem $(*)_\lambda$ is implicitly proved in the proof of [13, Theorem 1]:

THEOREM 5.2. *Let $\phi(x)$ and $\psi(x)$ be, respectively, a sub-solution and a super-solution of equation $(**)_\lambda$ such that $\phi(x) \leq \psi(x)$ on $\overline{D}$. If the function*

$$g_d(x, t) = (\lambda + d)t - h(x)t^p$$

*is monotonically increasing in* $t$, *that is, if we have*

$$g_d(x,s) < g_d(x,t) \qquad \text{for all } x \in \overline{D} \quad \text{and} \quad 0 \le s < t \le \|\psi\|_\infty,$$

*then equation* $(**)_\lambda$ *has a fixed point* $u(x) \in C(\overline{D})$ *such that*

$$\phi(x) \le u(x) \le \psi(x) \qquad \text{on } \overline{D}.$$

*In this case, the function* $u(x)$ *is a solution of problem* $(*)_\lambda$ *in the space* $C^{2+\theta}(\overline{D})$.

(III) We construct a sub-solution of equation $(**)_\lambda$. To do so, we need the following elementary results:

LEMMA 5.1. *If* $r_1 > r_2 \ge 1$, *then the first eigenvalue* $\lambda_1(D_r)$ *of problem* (5.3) *and its associated eigenfunction* $\phi_r(x)$ *satisfy, respectively, the conditions*

$$\lambda_1(D_{r_1}) = \left(\frac{r_2}{r_1}\right)^2 \lambda_1(D_{r_2}) \tag{5.6}$$

$$\phi_{r_1}(x) = \phi_{r_2}\left(\frac{r_2}{r_1}x\right), \qquad x \in D_{r_1}. \tag{5.7}$$

Since the eigenvalue $\lambda_1(D_r)$ depends continuously on $r$, it follows that, for each $\lambda < \lambda_1(D_0(h))$ close to $\lambda_1(D_0(h))$ there exists a constant $r > 1$ such that

$$\lambda = \lambda_1(D_r).$$

If we let

$$\lambda' = \lambda_1(D_{2r-1}),$$

then we obtain that its associated eigenfunction $\phi_{2r-1}(x)$ satisfies the conditions

$$\begin{aligned} (-\Delta + d)(\varepsilon\phi_{2r-1}) &\le (\lambda + d)\varepsilon\phi_{2r-1} - h(x)(\varepsilon\phi_{2r-1})^p \qquad \text{in } D_{2r-1} \\ \varepsilon\phi_{2r-1} &= 0 \qquad \text{on } \partial D_{2r-1}, \end{aligned} \tag{5.8}$$

if $\varepsilon$ may be chosen to be so small that

$$0 < \varepsilon \le \left(\frac{\lambda - \lambda'}{\sup_{D_{2r-1}} h}\right)^{1/(p-1)}. \tag{5.9}$$

However, using condition (5.2) and formula (5.6) we can prove that

$$\left(\frac{\lambda - \lambda'}{\sup_{D_{2r-1}} h}\right)^{1/(p-1)} \geq \left\{\frac{\lambda_1(D_r)(3r-1)}{2^{\sigma} C_1 (2r-1)^2}\right\}^{1/(p-1)} (r-1)^{(1-\sigma)/(p-1)}.$$

This implies that there exists a constant $\tilde{C} > 0$, independent of $r$ close to 1, such that condition (5.9) is valid for

$$\varepsilon = \tilde{C}(r-1)^{(1-\sigma)/(p-1)}.$$

Now we define a continuous function $v_r(x) \in C(\overline{D})$ as

$$v_r(x) = \begin{cases} \tilde{C}(r-1)^{(1-\sigma)/(p-1)} \phi_{2r-1}(x) & \text{in } D_{2r-1} \\ 0 & \text{on } \overline{D} \setminus D_{2r-1}. \end{cases}$$

Then, by assertion (5.8) it is easy to see that the function $v_r(x)$ is a sub-solution of equation $(**)_\lambda$. In fact, we have the following:

LEMMA 5.2. *There exists a constant $d_1 > 0$ such that, for all $d > d_1$ the function $v_r(x)$ satisfies the condition*

$$v_r(x) \leq K_d((\lambda + d)v_r - h v_r^p) \qquad \text{on } \overline{D}.$$

(IV) *End of Proof of Theorem* 5.1. First, by [5, Theorem 3.2] it follows that there exists a super-solution $w_\lambda(x) \in C^2(\overline{D})$ of problem $(*)_\lambda$ for each $\lambda_1 < \lambda < \lambda_1(D_0(h))$. Then we remark that the functions $Rw_\lambda(x)$ are super-solutions of equation $(**)_\lambda$ for all $R > 1$. Moreover, we can choose constants $R_0 > 1$ and $d > d_1$ so large that

$$v_r(x) \leq R_0 w_\lambda(x) \qquad \text{on } \overline{D}$$

and

$$g_d(x, s) < g_d(x, t) \qquad \text{for all } x \in \overline{D} \quad \text{and} \quad 0 \leq s < t < R_0 \|w_\lambda\|_\infty.$$

Hence it follows from an application of Theorem 5.2 that problem $(*)_\lambda$ has a solution $u(x) \in C^{2+\theta}(\overline{D})$ such that

$$v_r(x) \leq u(x) \leq R_0 w_\lambda(x) \qquad \text{on } \overline{D}.$$

However, by the uniqueness theorem for problem $(*)_\lambda$ (Theorem 1) we obtain that $u(x) = u_\lambda(x)$ in $D$, so that

$$v_r(x) = \tilde{C}(r-1)^{(1-\sigma)/(p-1)} \phi_{2r-1}(x) \leq u_\lambda(x), \qquad x \in D_{2r-1}. \quad (5.10)$$

Furthermore, we have, by formula (5.6),

$$r - 1 = \frac{(2r-1)^2}{3r-1}\left(\frac{\lambda_1(D_r) - \lambda_1(D_{2r-1})}{\lambda_1(D_r)}\right)$$
$$= \frac{r^2(2r-1)}{r+1}\left(\frac{\lambda_1(D_0(h)) - \lambda}{\lambda_1(D_0(h))}\right).$$

Summing up, we can rewrite inequality (5.10) in the form

$$C\big(\lambda_1(D_0(h)) - \lambda\big)^{(1-\sigma)/(p-1)}\phi_{2r-1}(x) \le u_\lambda(x), \qquad x \in D_{2r-1}, \quad (5.11)$$

where $C$ is a positive constant independent of $\lambda$.

On the other hand, by formula (5.7) it follows that, as $r \downarrow 1$,

$$\phi_{2r-1}(x) \to \phi_1(x) \qquad \text{in } C\big(\overline{D_0(h)}\big). \qquad (5.12)$$

Therefore the desired assertion (5.4) follows by combining inequality (5.11) and assertion (5.12).

Now the proof of Theorem 5.1 is complete. ∎

## REFERENCES


1. S. Alama and G. Tarantello, On the solvability of a semilinear elliptic equation via an associated eigenvalue problem, *Math. Z.* **221** (1996), 467–497.
2. H. Amann, Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, *SIAM Rev.* **18** (1976), 620–709.
3. M. A. del Pino, Positive solutions of a semilinear equation on a compact manifold, *Nonlinear Anal.* **22** (1994), 1423–1430.
4. J. M. Fraile, P. Koch Medina, J. López-Gómez, and S. Merino, Elliptic eigenvalue problems and unbounded continua of positive solutions of a semilinear elliptic equation, *J. Differential Equations* **127** (1996), 295–319.
5. J. L. Gámez, Sub- and super-solutions in bifurcation problems, *Nonlinear Anal.* **28** (1997), 625–632.
6. P. Korman and T. C. Ouyang, Exact multiplicity results for two classes of boundary value problems, *Differential Integral Equations* **6** (1993), 1507–1517.
7. J. Lee and T. Parker, The Yamabe problem, *Bull. Amer. Math. Soc.* **17** (1987), 37–91.
8. T. C. Ouyang, On the positive solutions of semilinear equations $\Delta u + \lambda u - hu^p = 0$ on the compact manifolds, *Trans. Amer. Math. Soc.* **331** (1992), 503–527.
9. T. C. Ouyang, On the positive solutions of semilinear equations $\Delta u + \lambda u + hu^p = 0$ on compact manifolds, Part II, *Indiana Univ. Math. J.* **40** (1991), 1083–1141.
10. C. V. Pao, "Nonlinear Parabolic and Elliptic Equations," Plenum, New York/London, 1992.

11. K. Taira, Bifurcation for nonlinear elliptic boundary value problems I, *Collect. Math.* **47** (1996), 207–229.
12. K. Taira and K. Umezu, Bifurcation for nonlinear elliptic boundary value problems II, *Tokyo J. Math.* **19** (1996), 387–396.
13. K. Taira and K. Umezu, Bifurcation for nonlinear elliptic boundary value problems III, *Adv. Differential Equations* **1** (1996), 709–727.
14. K. Taira and K. Umezu, Positive solutions of sublinear elliptic boundary value problems, *Nonlinear Anal.* **29** (1997), 761–771.